B.sc(H) part 3 paper 6

Topic:Normalizer(group theory)

subject:mathematics

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## Normlizer of an element of a group

**Defination**: *The set of all those elements of a group G which commute with a fixed element of a group G is called normaliser of an element a of the group and is denoted by N (a)*

Hence $N(a) = \{x \in G : ax = xa\}$

**Theorem**. *The normalizer $N(a)$ of an element $a \in G$ is a sub group of G.*

**Proof** : *We are going to prove this theorem taking Particular Cases.*

*Case (i) Normalizer $N(e)$ of element e of G is G itself.*

$$N(e) = \{x \in G : xe = ex\} = G$$

as $\quad xe = ex \ \forall \ x \in G.$

*Case (ii) Normalizer $N(a)$ of an element a of an abelian group is G itself.*

$$N(a) = \{x \in G : xa = ax\} = G$$

as $\quad xa = ax \ \forall \ x \in G$ when $G$ is abelian.

*Case (iii) Normalizer $N(a)$ is not necessarily a normal sub-group.*

We have know that $N(a)$ is a sub-group and let us denote it by $H$ i.e., $N(a) = H$.

If $h \in N(a)$ i.e., $H$, then $ha = ah$. ......(1

Now $N(a)$ or $H$ will be normal sub group if

$$xhx^{-1} \in H \text{ i.e. } N(a) \text{ for every } x \in G \text{ and } h \in H.$$

Further we must establish that

$$(xhx^{-1})\, a = a\, (xhx^{-1})$$

which is not necessarily true if the group be abelian for in that case

$$(xhx^{-1})\, a = xh\,(x^{-1}a) = xh\,(ax^{-1}) = x\,(ha)\,x^{-1}$$

$$= x\,(ah)\,x^{-1} = (xa)\,hx^{-1} = (ax\,hx^{-1}) = a\,(xhx^{-1}).$$

*Case (iv) Cyclic sub group of G generated by $a \in G$ is normal subgroup of the normalizer $N(a)$.*

$$N(a) = \{x \in G : xa = ax\}.$$

Let $H = \{a\}$ so that $h \in H$ is $a^n$.

Now $a^n a = a^{n+1} = a.a^n \quad \therefore \ a^n$ i.e $h \in N(a)$.

Since $\ h \in H \Rightarrow h \in N(a) \quad \therefore \ H$ is a sub group of $N(a)$

We have now to show that $H$ is a normal sub group of $N(a)$.

Let $h \in H$ be $a^n$ and $x$ be any arbitrary element of $N(a)$ so that

$$xa = ax.$$

$$\therefore \quad xhx^{-1} = xa^nx^{-1} = (x\,ax^{-1})^n = (ax\,x^{-1})^n$$

$$= (ae)^n = a^n \in H.$$

Thus $xhx^{-1} \in H \ \forall \ h \in H$ and $\forall \ x \in N(a)$ and hence $H$ is a normal sub group of $N(a)$.

Theorem ***Let a be any element of a group G. The two lements. x, y ∈ G give rise to the same conjugate of a if and nly if they belong to the same right coset of the normalizer V (a) of a in G.***

**Proof :** We shall show that if $x$ and $y$ belong to the same right coset f sub group $N(a)$ of $a$ in $G$ then they give the same conjugate of $a$ whereas, if they belong to different right cosets of sub group $N(a)$ of $a$ in $G$ hen they give different conjugates of $a$.

We have $Ha = Hb \Leftrightarrow ab^{-1} \in H$.

Now we know that $x \in Hx$, $y \in Hy$.

But if $x$, $y$ belong to the same right coset $Hx$ of sub group $H$ in $G$ then $x \in Hx$, $y \in Hx$. But $y \in Hy$.

$\therefore$ $Hx$ and $Hy$ have $y$ common and as such they must be identical as the two right cosets are either disjoint or identical.

$\therefore$ $Hx = Hy$ and hence $xy^{-1} \in H$.

Now let $x, y \in$ to same right coset of sub group $N(a)$ in $G$.

Therefore $xy^{-1} \in$ to $N(a)$ as shown above.

$\therefore$ $a(xy^{-1}) = (xy^{-1})a$ by definition of $N(a)$

$\Rightarrow x^{-1}(axy^{-1})y = x^{-1}(xy^{-1}a)y$

$\Rightarrow x^{-1}ax = y^{-1}ay.$

Above relation shows that $x$ and $y$ give rise to the same conjugate of $a$.

**2nd part.** Now let us suppose that $x$, $y$ belong to different right cosets of $N(a)$ in $G$ then we shall show that they cannot give rise to same conjugate of $a$.

*i.e.* $\quad x^{-1}ax \neq y^{-1}ay$

If possible let $x^{-1}ax = y^{-1}ay$

$\therefore$ $x(x^{-1}ax)y^{-1} = x(y^{-1}ay)y^{-1}$

or, $\quad axy^{-1} = xy^{-1}a \Rightarrow xy^{-1} \in N(a).$

But $xy^{-1} \in N(a)$ implies that $x$, $y$ belong to the same right coset of $N(a)$ as shown in first part which is a contradiction to the supposition that $x$, $y$ belong to different right cosets of $N(a)$ in $G$.

Hence $x^{-1}ax \neq y^{-1}ay$ *i.e*, $x$, $y$ cannot give rise to same conjugate of $a$.